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NUMERICAL SOLUTION OF THE PLATE BENDING AND FREE VIBRATIONS PROBLEMS*

S.D. ALGAZIN and K.I. BABENKO

The problem of the bending and free vibrations of a clamped and edge-supported plate is considered. The proposed algorithm is the algorithm described in /1/, made specific for the case of the biharmonic equation. It does not have saturation /2/, i.e., its accuracy will be the higher, the smoother the solution. The program is constructed in such a manner that if the plate boundary is sufficiently smooth and given parametrically, then several of the first eigenvalues can be calculated and the bending problem can be solved. An illustration is presented of the eigenfrequency computation for an edge-supported plate whose boundary (an epitrochoid) has a curvature of the order of 10^3 at twelve points (the curvatures enter explicitly in the appropriate boundary condition). The first eigenfrequencies are calculated with 7 – 8 places after the decimal point. The solution is obtained because of the accurate method of discretization and the study of the structure of the appropriate finite-dimensional problem. This would permit execution of computations with a large number of points (up to 1230). A comparison is given with the results of computations of other authors for a circle and an ellipse /3 – 5/.

1. Algorithms are considered for the numerical solution of the boundary value problems (1.1) – (1.3), and (1.1), (1.2) and (1.4):

$$\Delta^2 u(z) = F(z), \quad z \in G \tag{1.1}$$

$$u|_{\partial G} = 0 \tag{1.2}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial G} = 0 \tag{1.3}$$

$$\left.\frac{\partial^2 u}{\partial n^2} + \nu\left(\frac{\partial^2 u}{\partial s^2} + \frac{1}{\rho}\frac{\partial u}{\partial n}\right)\right|_{\partial G} = 0 \tag{1.4}$$

Here G is a domain in the complex z-plane with a sufficiently smooth boundary ∂G, n is the unit vector of the external normal to ∂G, $\partial/\partial s$ denotes differentiation with respect to the arclength (the length is measured counter-clockwise), $1/\rho$ is the curvature of ∂G, and ν is a constant (the Poisson's ratio). The function $F(z)$ is either given, or has the form $F(z) = (Q(z) + \lambda P(z))u(z)$, where Q and P are certain functions, and we have an eigenvalue problem for the biharmonic equation in this case. In particular, for $Q \equiv 0$ and $P \equiv 1$ we obtain a plate free vibrations problem, where the natural frequency ω is related to the spectral parameter λ by the relationship $\sqrt{\lambda} = \omega\sqrt{d/D}$, d is the density, and D is the cylindrical stiffness. The boundary conditions (1.2) and (1.3) signify that the plate is clamped along the edge, while the boundary conditions (1.2) and (1.4) signify edge-support.

Let $z = \varphi(\zeta)$ be a function giving the conformal mapping of the unit radius circle into the domain G. Then in place of (1.1) – (1.4), we obtain the following relations in the plane ζ:

$$\Delta(|\varphi'(\zeta)|^{-2})\Delta u = |\varphi'(\zeta)|^2 f(\zeta), \quad \zeta = re^{i\theta}, \quad r < 1 \tag{1.5}$$

$$u|_{r=1} = 0 \tag{1.6}$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = 0 \tag{1.7}$$

$$\frac{\partial^2 u}{\partial r^2} + \left.\left(\nu + (\nu - 1)\operatorname{Re}\left(\zeta\frac{\varphi''(\zeta)}{\varphi'(\zeta)}\right)\frac{\partial u}{\partial r}\right)\right|_{r=1} = 0 \tag{1.8}$$

$$f(\zeta) = F(z(\zeta)), \quad q(\zeta) = Q(z(\zeta)), \quad p(\zeta) = P(z(\zeta))$$

The condition (1.6) is taken into account in the boundary condition (1.8), i.e., we set $\partial^2 u/\partial s^2 = 0$.

A priori information about the solution, its analyticity, should be used for successful discretization of the boundary value problems (1.5) – (1.8). To this end we invert the differential operator in the left side of relation (1.5) and we apply the interpolation formula (2)

*Prikl.Matem.Mekhan.,46,No.6,pp.1011-1015,1982 for a function of two variables in a circle from /1/. This procedure is presented in detail below.

2. Let us introduce some notation. Let $l_j(\zeta)$ be the fundamental functions of the above-mentioned interpolation, then for any continuous function $f(\zeta)$ given in a unit circle, we have the following relationship

$$f(\zeta)=\sum_{j=1}^{N_*} f_j l_j(\zeta)+R_{N_*}(\zeta_j;f) \tag{2.1}$$

where $f_j=f(\zeta_j)$ is the value of the function f at the j-th interpolation node, R_{N_*} is the error in the interpolation formula. The form of the function $l_j(\zeta)$ is described in detail in /1/. We just note that the interpolation nodes ζ_j lie on m circles at equal angles and there are $N\equiv 2n+1$ nodes on each circle. The radius of the ν-th circle r_ν is a positive zero of the Chebyshev polynomial T_{2m} of degree $2m$. Therefore, there are $N_*=mN$ nodes in the circle. We use the notation

$$H_j(\zeta)=-\int_{|\xi|\leqslant 1} K(\zeta,\xi)\,l_j(\xi)\,d\xi,\quad K(\zeta,\xi)=\frac{1}{2\pi}\ln\left|\frac{1-\zeta\bar\xi}{\zeta-\xi}\right| \tag{2.2}$$

where $K(\zeta,\xi)$ is the Green's function of the Dirichlet problem for the Laplace equation in a circle. If ζ runs through the interpolation nodes ζ_j, $j=1,\ldots,N_*$, we then obtain the matrix H of dimension $N_*\times N_*$, $H_{ij}=H_j(\zeta_i)$. Therefore, H is a matrix of the discrete Dirichlet problem for the Laplace equation in a circle /1/. It turns out that the matrix H has the following modular structure:

$$H=\|h_{ij}\|;\quad i,j=1,2,\ldots,m \tag{2.3}$$

where the matrices h_{ij} are symmetric circulants /6/ of dimension $N\times N$. The properties of such matrices are studied in /7/.

3. We return to the first Laplace operator in (1.5) and we obtain

$$\Delta u(\zeta)=|\varphi'(\zeta)|^2\int_{|\xi|\leqslant 1} K(\zeta,\xi)\,|\varphi'(\xi)|^2 f(\xi)\,d\xi+|\varphi'(\zeta)|^2\int_0^{2\pi} K_0(\zeta,\theta)\,v(e^{i\theta})\,d\theta\equiv S(\zeta) \tag{3.1}$$

where $K_0(\zeta,\theta)$ is the Poisson kernel, and $v(e^{i\theta})$ is an unknown function. Turning again to the Laplace operator in (3.1), by taking account of (1.6) we obtain the relationship

$$u(\zeta)=\int_{|\xi|\leqslant 1} K(\zeta,\xi)\,S(\xi)\,d\xi \tag{3.2}$$

We apply the interpolation formula (2.1) to the functions $S(\xi)$ and $|\varphi'(\xi)|^2 f(\xi)$, and we use trigonometric interpolation for v

$$v(e^{i\theta})=\frac{2}{N}\sum_{j=0}^{2n} D_n(\theta-\theta_j)\,v_j+\tau_n(\theta;v),\quad \theta_j=\frac{2\pi j}{N} \tag{3.3}$$

(D_n is the Dirichlet kernel, and τ_n is the error in the interpolation). We then obtain

$$u(\zeta)=\sum_j H_j(\zeta)\,z_j\sum_i H_{ji}z_i f_i-\sum_j H_j(\zeta)\,z_j\sum_{p=0}^{2n} H_p{}^\circ(\zeta_j)\,v_p+\delta_1(\zeta) \tag{3.4}$$

$$z_j=|\varphi'(\zeta_j)|^2,\quad v_p=v(e^{i\theta_p}),\quad \theta_p=\frac{2\pi p}{N}$$

$$H_p{}^\circ(\zeta)=\int_0^{2\pi} K_0(\zeta,\theta)\,D_n(\theta-\theta_p)\,d\theta$$

Here z_j is the value of the function $|\varphi'(\zeta)|^2$ at the j-th interpolation node, and δ_1 is the error. The last integral is evaluated analytically.

The unknown quantities $v_0,\ldots,v_{2n}$ enter into the relationship (3.4). We use the second boundary condition (1.7) or (1.8) for their determination. It is convenient to consider a rather more general problem. We let M denote the differential operator in the left side of the boundary condition by considering the first boundary condition to have the form of (1.6). We apply the differential operator M to (3.4), we set $\zeta=e^{i\theta}$ in the relationship obtained by considering that θ runs through the interpolation nodes θ_j (see (3.3)). Then we obtain a system of linear equations with matrices A and right side R to determine $v=(v_0\ldots v_{2n})'$ where

$$A = \| A_{pq} \|, A_{pq} = \sum_{i=1}^{N_*} H^1_{i,p} z_i H_q^\circ(\zeta_i), \quad p, q = 1, \ldots, N \tag{3.5}$$
$$H^1_{i,p} = M(H_i(\zeta))|_{\zeta = e^{i\theta_p}}$$
$$R = (R_0, \ldots, R_{2n})', \quad R_p = \sum_{i,j}^{N_*} H^1_{j,p} z_i H_{ij} z_j f_j + \delta_2$$

and δ_2 is the error. Let $C = A^{-1}$, then $v = CR$. We substitute this expression into (3.4) and let ζ run through the interpolation nodes within the circle in the relationship obtained. We consequently have the summary relationship

$$u = (B^2 - BEB)f + \delta \tag{3.6}$$

Here $u = (u(\zeta_1), \ldots, u(\zeta_{N_*}))'$ is the vector of values of the function $u(\zeta)$ at the nodes, f is the corresponding vector of the values of the right side of the biharmonic equation, $B = HZ$ is the matrix of the discrete Dirichlet problem for the Laplace equation in the domain G under consideration (δ is the error of discretization)

$$E = \| E_{ij} \|, \quad E_{ij} = \sum_{p=0}^{2n} H_p^\circ(\zeta_l) \sum_{q=0}^{2n} C_{qp} \sum_i H^1_{i,q} z_i \tag{3.7}$$
$$Z = \operatorname{diaq}(z_1, \ldots, z_{N_*}), \; z_j = |\varphi'(\zeta_j)|^2$$

Discarding the error δ in (3.6), we obtain an approximate finite-dimensional problem. Therefore, the solution of the plate bending problem reduces to multiplying the matrix $D = B^2 - BEB$ by a vector, and the eigenvalue (free vibrations) problem corresponds to the approximate finite-dimensional problem

$$u = (B^2 - BEB)(Q + \lambda P)$$
$$Q = \operatorname{diag}(q(\zeta_1), \ldots, q(\zeta_{N_*})), \quad P = \operatorname{diag}(p(\zeta_1), \ldots, p(\zeta_{N_*}))$$

where Q, P are diagonal matrices for which values of the appropriate functions q and p at the interpolation nodes are on the diagonals. For the free vibrations problem $Q \equiv 0, P \equiv I$, i.e., it reduces to evaluation of the eigenvalues of the matrix D.

We note that the form of the second boundary value problem is taken into account by the configuration of the matrix E, where if the second boundary condition $\Delta u|_{\partial G} = 0$, then $E \equiv 0$, while for the conditions (1.7), (1.8), etc. the array E should be constructed according to (3.7). We recall that the first boundary condition is fixed and is applied in the form (1.6).

In order to see the stability of the algorithm described for the evaluation of the matrix D, the question of the specificity number of the matrix A should be investigated. It can be shown that the procedure to determine the function $v(\theta)$ on the boundary of the circle reduces to a problem of solving an integral equation of the first kind where the difficulty of performing the solution is determined by the rate at which its eigenvalues tend to zero. The inequality

$$\frac{p_0}{2(k+1)} \leqslant \lambda_k \leqslant \frac{p_1}{2(k+1)}, \quad p_0 = \min_{|\zeta| \leqslant 1} |\varphi'(\zeta)|^2, \quad p_1 = \max_{|\zeta| \leqslant 1} |\varphi'(\zeta)|^2$$

is valid for the eigenvalues λ_k of the corresponding integral equation under the boundary condition (1.7).

Therefore, in this case the specificity number of the matrix A of dimension $N \times N$ is $\operatorname{cond} A \asymp N$ and depends on the domain G under consideration.

Thus, a loss of $O(N)$ symbols occurs upon inversion of the matrix A (see (3.5)) for the case of boundary condition (1.7). In practical computations the value of $\operatorname{cond} A$ would never exceed a quantity of the order of 10^2 for both boundary conditions (the maximal value of N for which the computations would be performed would be 41).

If the initial domain is a circle, then the structure of the matrix D is the same as the structure of the matrix $\dot{H}$ defined by (2.3).

In fact, we first consider the boundary condition (1.7). In this case the array $H^1_{i,p}$ is represented in the form of the modular matrix H_1 while the array $H_q^\circ(\zeta_i)$ can be represented in the form of the modular matrix H_0

$$H_1 = (d_1, \ldots, d_m), \; H_0 = (b_1, \ldots, b_m)' \tag{3.8}$$

where d_ν and b_ν are symmetric circulants of dimension $N \times N$. Taking this notation into account, we obtain that

$$A = H_1 Z H_0 = d_1 Z_1 b_1 + \ldots + d_m Z_m b_m \tag{3.9}$$

Here $Z = \mathrm{diag}(Z_1, \ldots, Z_m)$ is a block-diagonal matrix, where the matrices Z_ν are diagonal and contain the values $|\varphi'(\zeta)|^2$ on the ν-th circle on the diagonals. Furthermore, we have

$$E = H_0 C H_1 Z = (b_1, \ldots, b_m)' C (d_1 Z_1, \ldots, d_m Z_m) \qquad (3.10)$$

The array $H^1_{i,p}$ can be represented in the form of a block matrix

$$H_2 = (d_1^\circ + \psi d_1, \ldots, d_m^\circ + \psi d_m), \psi = \mathrm{diag}(\psi_0, \ldots, \psi_{2m})$$

$$\psi_j = \nu + (\nu - 1)\,\mathrm{Re}\left(\zeta \frac{\varphi''(\zeta)}{\varphi'(\zeta)}\right)\Big|_{\zeta = e^{i\theta_j}}, \quad \theta_j = \frac{2\pi j}{N}$$

for the boundary condition (1.8).

Here d_ν are the same matrices as in the formula (3.8) while d_ν° are certain other symmetric circulants of dimension $N \times N$. The formulas for the arrays A and E are obtained by replacing the matrix H_1 by H_2 in (3.9) and (3.10). For a circle $Z \equiv I$ and $\psi_j \equiv \nu$. Hence, the matrices A and C are symmetric circulants of dimension $N \times N$. Now the assertion formulated above follows from the properties of symmetric circulants /6/.

Therefore, for a circle the matrix D possesses properties formulated in /7/. It is known that in a circle the corresponding eigenvalue problem for the biharmonic equation is reduced by separation of variables to an eigenvalue problem for ordinary differential equations. The finite-dimentional problem inherit such property. The form of the eigenfunction and the multiplicity property of part of the eigenvalues is also inherited.

We note that the arrays H and H_0 which are required for evaluation of the matrix D are evaluated once for all the domains and boundary conditions of the kind under consideration. Moreover, the array H contains $m^2(n+1)$ distinct elements and the array H_0 just $m(n+1)$. For instance, if 104 points (eight thirteen-point circles) are taken in a circle, then the array H contains 448 distinct elements and H_0 has 56. The arrays H_1 and H_2 also have an analogous structure. This permits execution of computations with a large number of points.

The maximal number of points with which the computations were performed is 1230 (30 circles of 41 points). In practice, eigenvalues of large dimension matrices are evaluated by the simple iteration method in conjunction with the method of elimination /8/. The approximate expression of the eigenfunction obtained at the least number of points was used as the initial approximation. Only multiplication programs for the matrices D and D' (i.e., the transpose matrix) by a vector are required to realize this method, where the matrices D and D' need not be evaluated explicitly.

The simple matrix configuration described above for the finite-dimensional problem permits creation of a standard program to evaluate the eigenvalues of the biharmonic operator and to solve the corresponding boundary value problem.

4. Let us examine certain examples of numerical computations. We initially consider the problem of free vibrations in a circle. The maximum number of points with which the computations were performed is 820 (20 circles of 41 points). Evaluation of the eigenvalues of the corresponding matrix D of dimension 820 X 820 reduces to evaluation of 21 matrices of dimension 20 X 20 /7/. For the first five single eigenvalues we obtain for the clamping boundary condition 104.3631056 (104.344 /3/), 1581.744 (1581.306/3/), 7939.549, 25022.25, 61012, and for the free support boundary condition ($\nu = 0.25$) we obtain 23.62085804 (24.744 /3/), 879.843510932 (885.481 /3/), 5491.02409476, 19117.1544172); 49357.5252428. As a check, calculation were performed at 100 points by a one-dimensional method. Only the symbols that agree are presented above (except for the last digit). We present still another value of the seventh eigenvalue in magnitude for the free support case (double) 3224.568989 (3259.626 /3/).

As the next example, we consider the problem of free vibrations of a freely supported plate whose boundary is obtained from a circle by the conformal mapping $z = \zeta(1 + \zeta^{12}/14)$. The boundary of this domain (an epitrochoid) has a curvature of order 10^3 (− 2170) at 12 points. For the first five eigenvalues for $\nu = 0.25$, we obtain at the 1230 = 30 X 41 points 10.6434390885, 95.7918067'703; 272.3244532'59; 471.2710702'79; 587.1410694'72; Moreover, computations were performed at 410 = 10 X 41 and 820 = 20 X 41 points, whose results agree with the results presented above with an error in not more than the ninth unit of the last symbol, separated by the ordinary print from the subsequent cursive print.

This last illustration is the evaluation of the fundamental frequency for a clamped elliptic plate. The conformal mapping of the circle on the ellipse was executed numerically. For $\sqrt{\lambda_1}$ with $a = 1, b = 0.5$, the value 27.2 (28.5148 /4/; 27.5 /9/) is obtained at 104 = 8 X 13 points, while for $a = 1, b = 1/3$ the value 56.4 (60.3179 /4/, 56.9 /9/) is obtained at 410 = 10 x 41 points.

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